Master's thesis in biomathematics
Population dynamical embedding of iterated games of different lengths

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Tiivistelmä — Referat — Abstract

Topic:

-This thesis addresses the problem of comparing payoffs from iterated games of varying lengths in a meaningful way. Shorter games can be played more often than longer games in same amount of time. Direct comparison of payoffs per game therefore leads to systematic error for the shorter games. On the other hand if it is difficult to find a playing partner, shorter games have extra disadvantage and per game payoff calculation is more accurate. This thesis calculates payoffs as time averages instead of per game averages taking into account rate of finding a new playing partner.

-Games can be of different lengths because of random termination of the game or by a strategic choice of the player. Latter case is known as quitting strategy, which is given in the form of a quitting rule as a part of a players strategy, e.g. "quit after two subsequent rounds with low payoff". Quitting can prevent further losses in a single iterated game, but becomes more effective when a player can start a new game with another opponent after quitting in a game. Opponents are randomly chosen from a pool of potential players and after the termination of a game they are returned to the pool to be paired off randomly again. This is called "pooling". The strategies utilized by the players in the pool change over time as strategies with longer games become more rare in the pool.

-Quitting traditionally has not been considered a strategic choice.

Method:

This thesis constructs a model for iterated games with quitting and pooling. Then it is explored further with an example of iterated Hawk-Dove-Bully-Retaliator (HDBR) game.

Results:

- Strategies that tend to lead to long games become less frequent in the pool than strategies with shorter games.
- Greedy strategies, when pooled with quitting strategies, will eventually spend most of their time playing against each other or in the pool. This reduces their payoffs to the point that they are no longer competitive compared to more altruistic strategies.
- High termination rate increases the relevance of the first few rounds. This causes more greedy strategies to benefit from high termination rate when more naive or altruistic strategies cannot play in beneficial games for long.

Avainsanat — Nyckelord — Keywords

Replicator equation, discrete dynamical systems, matrix games, pooling, quitting strategy, Leslie matrix, resident-invader dynamics, iterated games, pure strategy, Hawk-Dove game, pairwise invasibility graph, ESS, game theory

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Contents

Introduction

Two by two matrix games have been studied by number of authors [1] [3] [5]. Matrix games give players option to play either option A or option B. Then each player is given payoff according to payoff matrix. Strategy in a game are fixed instructions on how to choose what to play. Replicator equation [2] and payoffs given to the strategies are used to see if a strategy will increase or decrease in density in the population of strategies.

In iterated games each strategy plays multiple rounds against same opponent. I add quitting, pooling and random game terminating event to the iterated games. Quitting is usually not considered a valid strategy, but in this thesis it is considered part of the strategy space. Quitting rule allows some strategies to leave unfavourable games before they lose too much. Pooling is a new construction that solves asynchronosity in iterated games and allows strategies to favour shorter games. Random game terminating event is usual to avoid infinite games and to take into account that no game lasts forever in reality.

Iterated games count payoff per game, which gives wrong estimation of shorter games payoffs. This thesis constructs a system called pooling to account for different length games where shorter games might be played more often than longer games. Payoffs are not calculated per game, but as time averages. Each strategy starts in a pool of strategies and are paired at fixed rate with other strategies to play a game at the end of each round. Each game that ends has its playing strategies send back to pool to be paired again. Pooled games allows strategies to quit quickly in detrimental games and to play against new opponent instead of being stuck with a bad opponent or waiting for longer games to end.

I will study examples of pooled games under Hawk-Dove-Retaliator-Bully game. These results will be presented with new graphical tool to conveniently see ESS between multiple strategies. I will also provide full classification of these Pairwise Invasibility Graphs (PIG) in the iterated HDRB-game with pooling and quitting rules.

Chapter 1

Pooled Games

1.1 Replicator equation

We start with defining replicator equation as system of differential equations

$$\dot{D}_i = D_i(f_i(D) - \phi(D)), \tag{1.1}$$

$$\phi(D) = \sum_{j=1}^{n} D_j f_j(D), \tag{1.2}$$

where D_i is proportion of type i strategy in population, vector $D = (D_1, D_2, \ldots, D_n)$ gives the distribution of types in the population, function $f_i(D)$ gives fitness of type i strategy against the population and function $\phi(D)$ is weighted average of population fitness. Additionally in this thesis we assume that total population size is always scaled to 1.

To define function $f(D)$, we need to define how fitness function is determined. System that determines values of function f has much faster dynamics than replicator equation and thus we assume that at each time τ of replicator equation, system determining fitness has already reached its equilibrium.

1.2 Quitting strategies and asynchronous games

Usually deliberately quitting is not considered a strategy in two-by-two matrix games, but here I assume that each strategy has some rule they follow to define if they quit or not. Rule may be as simple as 'never quit'. Quitting strategy should not depend on strategys opponent directly, only on what the opponent plays each round.

Since not all games and rounds within are not necessarily same length, it brings a problem of asynchronosity. I assume each round is roughly same length and each game starts same time as new rounds start, then rounds are synchronous enough that I do not need to think of half finished rounds at each step. In nature, we can assume each round takes roughly a day and nights are rest periods where no games are played. This means that rounds are asynchronous but each evening situation is always that at most one round has been played per game and new rounds do not start until next dawn. Other such periodic events might work for dividing rounds. See figures 1.1 and 1.2 for two ways iterated games normally calculate payoffs.

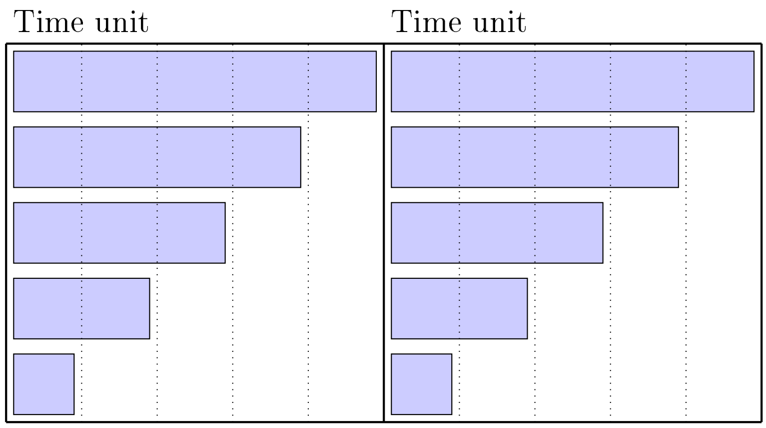

Figure 1.1: Usually payoff per player is calculated per game. In this graph each such time unit is shown as big box. Games are marked with blue boxes and dotted lines show how rounds are counted. As seen here, players with shorter games have a lot of idle time, waiting for others to finish their games. This usually leads to underestimation of viability of shorter games.

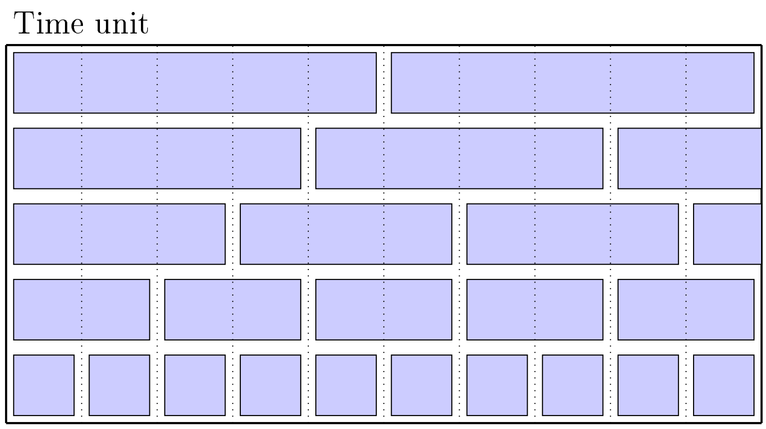

Figure 1.2: This figure shows one way to calculate payoff in iterated games. Each game is marked in blue and dotted lines show when each round ends. In this kind of system payoff is usually calculated per round. This way leads to overestimation of payoff from shorter games, if it is not guaranteed to immediately find a new game after previous ends.

1.3 Defining pooled games

Strategies start in pool and are paired with other strategies to play a game. After each round each game either ends or continues. If game continues, each strategy playing advances to next round. Strategies whose game ended are returned to the pool and from the pool new pairings are send to play new games. This system is then continued until equilibrium is reached (which will take infinite time).

Let $S_n(t)$ be a density of free strategy n at discrete time t, f_{ij} maximum number of rounds played between strategies i and j, $G^h_{ij}(t)$ density of games on round h between strategies i and j at time t. Let $p \in (0, 1)$ be pairing probability and $\delta \in (0, 1)$ probability that game continues after a round ends. Dynamical system is given by

$$G^n_{ij}(t+1) = \delta G^{n-1}_{ij}(t) \tag{1.3}$$

$$G^1_{ij}(t+1) = p\frac{S_i(t)S_j(t)}{\sum\limits_n S_n(t)} \tag{1.4}$$

$$S_i(t+1) = (1-p)S_i(t) + \sum_j G^{f_{ij}}_{ij}(t). \tag{1.5}$$

and the initial conditions are

$$\sum_{i,j,h} G^h_{ij}(0) = 0, \quad S_i(0) = D_i, \quad \sum_i S_i(0) = 1. \tag{1.6}$$

Assume that $t \to \infty$ and that system is at equilibrium, then $S_i(\infty) := S_i$. Then

$$pS_i = \sum_j G_{ij}^1 \tag{1.7}$$

$$S_i(0) = S_i + \sum_n \sum_j G_{ij}^n \tag{1.8}$$

$$\overset{1.3}{=} S_i + \sum_j \frac{1-\delta^{f_{ij}}}{1-\delta} G_{ij}^1 \tag{1.9}$$

$$\overset{1.4}{=} S_i + \sum_j \frac{1-\delta^{f_{ij}}}{1-\delta} p \frac{S_i S_j}{\sum_n S_n} \tag{1.10}$$

$$= S_i + pS_i \sum_j \frac{1-\delta^{f_{ij}}}{1-\delta} \frac{S_j}{\sum_n S_n} \tag{1.11}$$

$$\Rightarrow \qquad 1 = \frac{S_i}{S_i(0)}(1 + p\sum_j \frac{1-\delta^{f_{ij}}}{1-\delta} \frac{S_j}{\sum_n S_n}) \tag{1.12}$$

From this we can see that if strategy plays more games against itself (increasing f_{ii}), its density of non-playing strategies (S_i) must decrease. This means that other strategies will be paired more rarely with this strategy as the density of free strategies is lower. This sort of density manipulation by strategies does not happen without pooling and quitting.

1.4 Single strategy

With single strategy we have

$$1 = S_1(1 + p\frac{1-\delta^{f_{ij}}}{1-\delta}) \tag{1.13}$$

and corresponding matrix for round to round flow is

(1.14)

$$P := \begin{bmatrix} 1-p & 1-\delta & 1-\delta & 1-\delta & \dots & 1-\delta & 1 \\ p & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \delta & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \delta & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \delta & 0 \end{bmatrix}, \qquad \begin{pmatrix} S_i \\ G_{ii}^1 \\ \vdots \\ G_{ii}^{f_{ii}} \end{pmatrix} (t+1) = P \times \begin{pmatrix} S_i \\ G_{ii}^1 \\ \vdots \\ G_{ii}^{f_{ii}} \end{pmatrix} (t)$$

which is a Leslie matrix. As nothing is added or removed from the system, means that leading eigenvalue $\lambda_1 = 1$ and eigenvector corresponding to stable distribution is

$$(1, p, \delta p, \delta^2 p, \ldots, \delta^{f_{ii}} p)^\top. \tag{1.15}$$

1.5 Two strategies

Define $x := S_1/S_2$ and $x_0 := S_1(0)/S_2(0)$, now from equation 1.11 we get

$$x_0 = \frac{S_1(0)}{S_2(0)} = \frac{S_1 + \sum\limits_j \sum\limits_{h=1}^{f_{1j}} G_{1j}^h}{S_2 + \sum\limits_j \sum\limits_{h=1}^{f_{2j}} G_{2j}^h} = \frac{S_1 + \sum\limits_j G_{1j}^1 \sum\limits_{h=1}^{f_{1j}} \delta^{h-1}}{S_2 + \sum\limits_j G_{2j}^1 \sum\limits_{h=1}^{f_{2j}} \delta^{h-1}} \tag{1.16}$$

$$= \frac{S_1 \sum\limits_j S_j + S_1 \sum\limits_j p S_j \sum\limits_{h=1}^{f_{1j}} \delta^{h-1}}{S_2 \sum\limits_j S_j + S_2 \sum\limits_j p S_j \sum\limits_{h=1}^{f_{2j}} \delta^{h-1}} = \frac{S_1 \sum\limits_j S_j (1 + p\frac{\delta^{f_{1j}}-1}{\delta - 1})}{S_2 \sum\limits_j S_j (1 + p\frac{1-\delta^{f_{2j}}}{1-\delta})} \tag{1.17}$$

$$= \frac{1/S_2}{1/S_1} \frac{\sum\limits_j S_j (1 + p\frac{1-\delta^{f_{1j}}}{1-\delta})}{\sum\limits_j S_j (1 + p\frac{1-\delta^{f_{2j}}}{1-\delta})} = \frac{x(1 + p\frac{1-\delta^{f_{11}}}{1-\delta}) + (1 + p\frac{1-\delta^{f_{12}}}{1-\delta})}{(1 + p\frac{1-\delta^{f_{21}}}{1-\delta}) + x^{-1}(1 + p\frac{1-\delta^{f_{22}}}{1-\delta})}. \tag{1.18}$$

Assuming strategy 1 is the resident and strategy 2 is an invading strategy, we calculate from 1.18

$$\lim_{x_0 \to \infty} \frac{dx}{dx_0} = \frac{(1 + p\frac{1-\delta^{f_{12}}}{1-\delta})}{(1 + p\frac{1-\delta^{f_{11}}}{1-\delta})} \tag{1.19}$$

which shows that when x_0 is large, x is also large and of same magnitude. This means that initially rare strategy in the population will be rare during the payoff calculations. This also indicates that game lengths with different strategies affect how their sizes in population are accounted for. Knowing x, it is possible to calculate equilibrium mean per capita payoff to strategy i.

Let c_{ij}^h be payoff to strategy i against strategy j on round h. Now We can calculate per capita payoff to strategy i against the starting pool of strategies to be

$$E_i = \frac{\sum\limits_j \sum\limits_{h=1}^{f_{ij}} G_{ij}^h c_{ij}^h}{S_i(0)} = \frac{\sum\limits_j \sum\limits_{h=1}^{f_{ij}} p\delta^{h-1} \frac{S_i S_j}{\sum_n S_n} c_{ij}^h}{S_i(0)} = \frac{pS_1}{S_1(0)\sum\limits_n S_n} \sum_j \sum_{h=1}^{f_{ij}} \delta^{h-1} S_j c_{ij}^h \tag{1.20}$$

$$= \frac{pS_1}{S_1(0)\sum\limits_n S_n} \sum_j S_j \sum_{h=1}^{f_{ij}} \delta^{h-1} c_{ij}^h = \frac{pS_1}{S_1(0)\sum\limits_n S_n} \sum_j S_j \sum_{h=1}^{f_{ij}} a_{ij}^h, \tag{1.21}$$

where $a_{ij}^h = \delta^{h-1} c_{ij}^h$. Using this we can calculate fitness for strategy 2 against itself and strategy 1. Payoff against population mean in mixed population of strategies 1 and 2 to strategy 2 is

$$E_2 - E_{1,2} = \frac{p}{S_1 + S_2}\left(\frac{S_1 S_2}{S_2(0)} \sum_h^{f_{21}} a_{21}^h + \frac{S_2^2}{S_2(0)} \sum_h^{f_{22}} a_{22}^h - \frac{S_1^2}{S_1(0)} \sum_h^{f_{11}} a_{11}^h - \frac{S_1 S_2}{S_1(0)} \sum_h^{f_{12}} a_{12}^h\right) \tag{1.22}$$

$$= p\frac{S_2}{S_2(0)} \frac{S_2}{S_1 + S_2} p(x \sum_h^{f_{21}} a_{21}^h + \sum_h^{f_{22}} a_{22}^h - \frac{x_2}{x_0} \sum_h^{f_{11}} a_{11}^h - \frac{x}{x_0} \sum_h^{f_{12}} a_{12}^h) \tag{1.23}$$

$$= p\frac{S_2}{S_2(0)} \frac{1}{1 + x}(x(\sum_h^{f_{21}} a_{21}^h - \frac{x}{x_0} \sum_h^{f_{11}} a_{11}^h) + (\sum_h^{f_{22}} a_{22}^h - \frac{x}{x_0} \sum_h^{f_{12}} a_{12}^h)) \tag{1.24}$$

We know that x/x_0 converge to a constant when $x_0 \to \infty$. From this we can conclude that if strategy 1 is the resident ($x_0 \to \infty$) only games against the resident strategy matter. Also it is important to note that without quitting $x/x_0 = 1$. When $p \to 0$ we are starting new games only when all the other games have ended. This means that $x/x_0 = 1$ and we are back to regular invasion fitness of single game in two by two matrix game. If $\delta = 1$ we require quitting strategy or each strategy will only play in games they are first assigned to and only repeated rounds matter. Opposite end is when $\delta = 0$ where every strategy is pure strategy as no second round of a game is ever played.

Chapter 2

Single round HDRB game

2.1 Hawk-Dove-Retaliator-Bully game

We consider traditional Hawk-Dove game where each strategy chooses to play either Hawk or Dove. Let $C > 0$ be the cost of fighting and $V > 0$ be the reward. Hawk versus Hawk is seen as one is injured and pays the cost, while other takes the reward, multiplying both by half gives us the average result. If Dove and Hawk meet, Dove gets nothing and Hawk takes whole reward. When two Doves meet, they share the reward or each get it with equal probability. [4] See figure 2.1. In single round Hawk-Dove game there are strategies called Bully and Retaliator. Bully plays hawk against Dove but folds against Hawk and plays dove. Bully against itself chooses hawk. Retaliator plays dove against Dove and Retaliator strategies but plays hawk versus Bullies and Hawks.

	meets Hawk	meets Dove	meets Bully	meets Retaliator
Hawk	V/2 - C/2	V	V	V/2-C/2
Dove	0	V/2	0	V/2
Bully	0	V	V/2 - C/2	0
Retaliator	V/2 - C/2	V/2	V	V/2

Figure 2.1: Single round game of Hawk and Dove payoffs to all four single round game strategies.

To calculate Hawk vs Dove game payoffs, let h be density of Hawk players and d be density of Dove players with $h + d = 1$. Payoff to a hawk player is

$$\frac{V-C}{2}h + Vd = \frac{1}{2}(V - C + d(V+C))$$

and to dove player it is

$$0h + \frac{V}{2}d = \frac{dV}{2}.$$

When they are at equilibrium, we know that they are equally viable, assuming solution for d, if it is smaller than 0 dove cannot invade, as payoff for dove players is less than for the hawk players for all densities. Same reasoning for hawk, if $d \geq 1$ hawk cannot invade. Assuming two payoffs are equal and $d, h \in (0,1)$ we get condition

$$\begin{aligned} dV &= V - C + d(V+C) \\ d &= \frac{C-V}{C} = 1 - \frac{V}{C} \end{aligned}$$

this gives us condition for coexistence

$$\begin{aligned} 1 - \frac{V}{C} &> 0 \\ C &> V. \end{aligned}$$

Bully versus Dove strategy is same as with Hawk versus Dove. Assuming similarly to Dove versus Hawk calculations that $b + h = 1$, where b is density of Bully strategies and h is density of Hawk strategies, equilibrium is at $h = 1$ when

$$\begin{aligned} h\frac{V-C}{2} + bV &= b\frac{V-C}{2} \\ hV - hC + 2V - 2hV &= V - C - hV + hC \\ -hC + V &= -C + hC \\ h &= \frac{1}{2}(\frac{V}{C} + 1) > 1 \\ V &> C. \end{aligned}$$

Lets indicate density of Hawk strategies again with $h \in (0, 1)$. Now we can see that payoff to Retaliator strategy in mixed population is always larger than to Hawk. Calculating payoff gives

$$\begin{aligned} \frac{V-C}{2} &< h\frac{V-C}{2} + (1-h)\frac{V}{2} \\ V - C &< hV - hC + V - hV \\ -C &< -hC \\ h &< 1. \end{aligned}$$

Retaliator strategy versus Bully strategy gives us that retaliator strategy always has higher payoff. Assume $r \in (0, 1)$ is the density of retaliator strategy, then

$$\begin{aligned} r\frac{V}{2} + (1-r)V &> (1-r)\frac{V-C}{2} \\ rV + 2V - 2rV &> V - C - rV + rC \\ V &> C(r-1). \end{aligned}$$

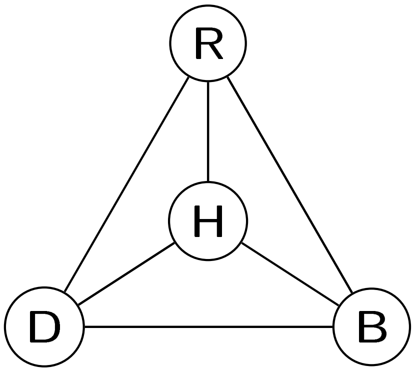

Figure 2.2: P.I.G. without any invasion data.

2.2 Pairwise Invasibility Graph

Pairwise Invasibility Graphs or PIGs show quickly which strategy can invade which strategy. See figures 2.2, 2.3, 2.4, 2.5 and 2.6. Each node in the graph corresponds to a strategy and arrows indicate which strategies they can invade and which strategies can invade them.

In figure 2.7 and in figure 2.8 Retaliator strategy is ESS, it cannot be invaded by any of the other strategies. ESS is marked blue in each plot for clarity. Line between Dove and Retaliator is dotted because neither have edge over another. Interesting part is the triangle of Hawk, Dove and Bully strategies. When reward V is larger than cost of competing C Hawk can invade the other strategies and it cannot be invaded by two others. In this situation selfishness is clearly the best option. However when cost of competing goes up, every strategy (Dove, Hawk, Bully) can invade each other and coexist.

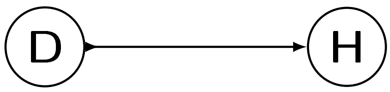

Figure 2.3: Hawk can Invade, Dove cannot invade.

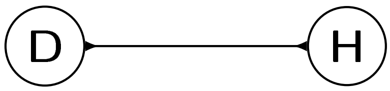

Figure 2.4: Both can invade, leading to coexistence.

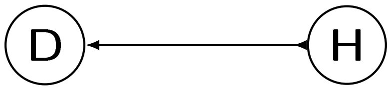

Figure 2.5: Hawk cannot invade, Dove can invade.

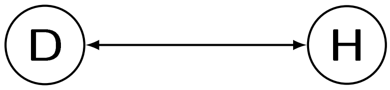

Figure 2.6: Neither can invade the other, mutual exclusion.

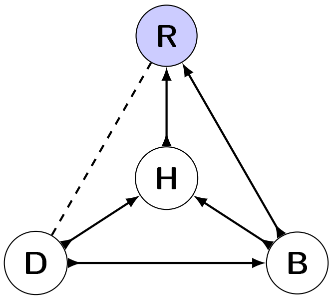

Figure 2.7: Single round game pairwise invasibility graph for 0<C<V.

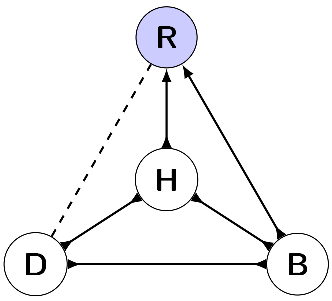

Figure 2.8: Single round game pairwise invasibility graph for 0<V<C.

Chapter 3

Iterated HDRB game

3.1 PIGs and Pooled games

Next I build pairwise invasibility graphs (PIG) for Dove, Hawk, Retaliator and Bully strategies in iterated games with pooling and quitting (pooled games). I assume Dove and Hawk quit immediately after a Hawk strategy is played against them. Retaliator and Bully do not quit the game. Dove as a game strategy is a strategy that always chooses to play a Dove. Hawk likewise with Hawk. Retaliator game strategy is one that starts playing Dove and copies whatever opponent played last round. Bully is a game strategy that starts with Hawk, but switches to Dove if opponent plays a Hawk strategy against them. It does not switch back to Hawk. When strategies face each other, if one strategy would quit, game ends. For each pairing see figure 3.1. Retaliator and Bully play differently from single round game, but the behaviour is similar, retaliator escalates if opponent does and bully starts with hawk but loses its will to fight if facing hawk.

f	meets Hawk	meets Dove	meets Retaliator	meets Bully
Hawk	1	1	2	1
Dove	1	1	∞	1
Retaliator	2	∞	∞	∞
Bully	1	1	∞	∞

Figure 3.1: Game lengths in each match up.

3.2 Hawk and Dove invasibility

Lets assume that Dove is the resident strategy and Hawk is the invading strategy. As resident $S_d(0) \approx 1$ and invading $S_h(0)$ is small, now mark number of average number of games as

$$
\begin{aligned}
r(i,j) &= \frac{1-\delta^{f_{ij}}}{1-\delta} \\
E_h - E_d &= V r(h,d) - \frac{V}{2}\frac{1+p\ r(h,d)}{1+p\ r(d,d)} r(d,d) \\
&= \frac{V}{2(1+p\ r(d,d))}(2r(h,d) + 2p\ r(h,d)r(d,d) - r(d,d) - p\ r(d,d)r(h,d))
\end{aligned}
$$

We know from before that $f_{dh} = f_{hh} = 1$ and $f_{dd} \to \infty$. Now we have condition for hawk invasion as

$$
\begin{aligned}
\Rightarrow \qquad & 2 + 2p\frac{1}{1-\delta} - \frac{1}{1-\delta} - \frac{p}{1-\delta} > 0 \\
\Rightarrow \qquad & 2 - 2\delta + 2p - 1 - p > 0 \\
\Rightarrow \qquad & \delta < \frac{1+p}{2}.
\end{aligned}
$$

This means that if games end quickly due to random event and large portion of strategies are paired each step, hawk can invade. However for each pairing rate p exists $\delta \in (\frac{1+p}{2}, 1)$ such that Hawk strategy cannot invade, regardless of reward and punishment values of C and V.

Now lets assume Hawk is resident strategy and dove is the invader. Using notation from above and $S_h(0) \approx 1$ and $S_d(0)$ is small, we get

$$
\begin{aligned}
E_d - E_h &= -\frac{1+p\ r(h,d)}{1+p\ r(h,h)}\frac{V-C}{2} r(h,h) > 0 \\
&\Rightarrow C > V,
\end{aligned}
$$

which is the what is in single games used to prevent Hawk being the only viable strategy. Additionally in single games Hawk can always invade Dove, which is not the case in pooled games. In figures 3.2 and 3.3 pairwise invasibility is shown in parameter space $\{p, \delta\}$.

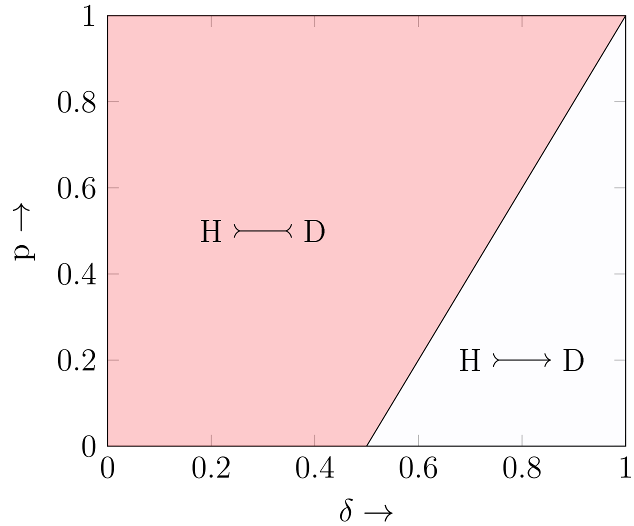

Figure 3.2: Hawk and Dove invasion when $C > V$

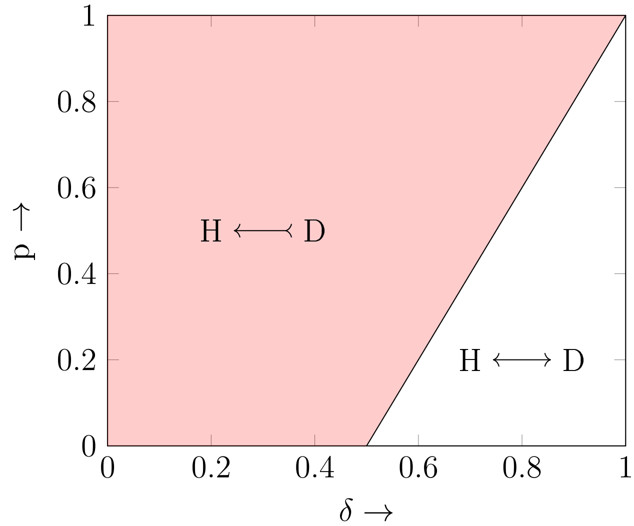

Figure 3.3: Hawk and Dove invasion when $C < V$

3.3 Hawk and Bully invasibility

Now Hawk is the resident strategy and Bully is a invading strategy. Now $S_h(0) \approx 1$ and $S_b(0)$ is small. Condition for Bully invasion is

$$E_b - E_h = \frac{V-C}{2} - \frac{1+p\ r(h,b)}{1+p\ r(h,h)}\frac{V-C}{2} = 0.$$

This means that the first ESS condition does not tell if Bully can invade or not. To find out if Bully can invade, we need to check second ESS condition. This means that Bully invades Hawk population if Hawk cannot invade Bully.

Now assume Bully as resident Strategy and Hawk as the invading strategy. Now $S_b(0) \approx 1$ and invading $S_h(0)$ is small. Invasion criteria is

$$\begin{aligned} E_h - E_b &= \frac{V-C}{2} - \frac{1+p\ r(h,b)}{1+p\ r(b,b)}\left(\frac{V-C}{2} + \delta r(b,b)\frac{V}{2}\right) > 0 \\ &\Rightarrow - p\delta C - \delta V > 0, \end{aligned}$$

which cannot be satisfied. This means that Hawk cannot invade Bully and thus Bully can always invade Hawk.

3.4 Dove and Bully invasibility

Set Dove as resident strategy and Bully as invading strategy. Now $S_d(0) \approx 1$ and invading $S_b(0)$ is small. Bully invasion criteria is

$$\begin{aligned} E_b - E_d = V - \frac{1+p\ r(d,b)}{1+p\ r(d,d)}\left(\frac{V}{2}r(d,d)\right) &> 0 \\ V\left(1+\frac{p}{1-\delta}\right) - (1+p)\left(\frac{V}{2}\frac{1}{1-\delta}\right) &> 0 \\ \frac{1}{2} + \frac{p}{2} - \delta &> 0 \\ \delta &< \frac{1+p}{2}, \end{aligned}$$

which is the same as for Hawk invasion into Dove population.

Now assume Bully as resident strategy and Dove as the invading strategy. Now $S_b(0) \approx 1$ and invading $S_d(0)$ is small. Invasion criteria is

$$E_d - E_b = 0 - \frac{1 + p\ r(d,b)}{1 + p\ r(b,b)}(\frac{V-C}{2} + \frac{V}{2}\frac{\delta}{1-\delta}) > 0$$

$$V - C + \frac{\delta V}{1-\delta} < 0$$

$$\delta < 1 - \frac{V}{C}.$$

Figure 3.4 shows that all possible invasion combinations are possible between the Bully and the Dove strategies in parameter space $\{p, \delta\}$.

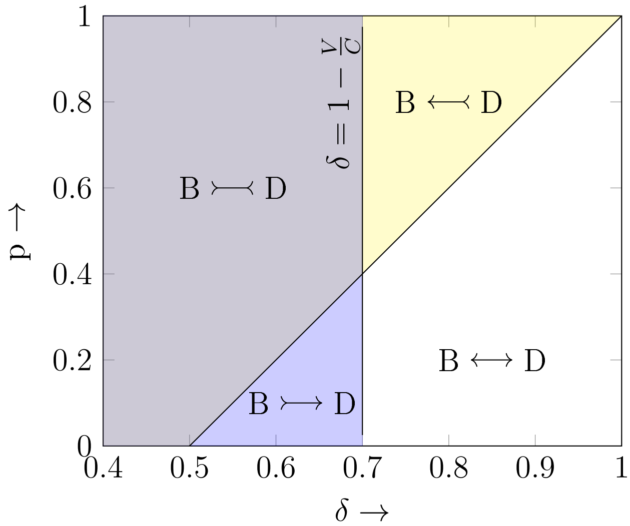

Figure 3.4: Bully and Dove pairwise Invasibility when C > 2V

3.5 Dove and Retaliator invasibility

Dove and Retaliator both only play dove and each plays for as long as possible, meaning they are indistinguishable when playing against each other. As neither fills invasion criteria they cannot invade each other. However they are not mutually exclusive either.

3.6 Hawk and Retaliator invasibility

Set Hawk as resident strategy and Retaliator as invading strategy. Now $S_h(0) \approx 1$ and invading $S_r(0)$ is small. Hawk invasion criteria is

$$
\begin{aligned}
E_r - E_h = 0 + \delta\frac{V-C}{2} - \frac{1+p\,r(h,r)}{1+p\,r(h,h)}\frac{V-C}{2} > 0 \\
\delta - p < 1,
\end{aligned}
$$

which is always true for $\delta, p \in (0,1)$.

Now assume Retaliator as resident strategy and Hawk as the invading strategy. Now $S_r(0) \approx 1$ and invading $S_h(0)$ is small. Invasion criteria is

$$
\begin{aligned}
E_h - E_r = V + \delta\frac{V-C}{2} - \frac{1+p\,r(h,r)}{1+p\,r(r,r)}\frac{V}{2}r(r,r) > 0 \\
Vp - \delta pC + V - \delta C - \delta V - \delta^2 V + \delta^2 C > 0 \\
p(V-\delta C) > -V + \delta(C+V) + \delta^2(V-C)
\end{aligned}
$$

at the extreme values of δ we have

$$
\begin{aligned}
p(V-C) > 3V, \text{ when } \delta = 1, \text{ which is false } \forall p \\
pV > -V, \text{ when } \delta = 0, \text{ which is true } \forall p
\end{aligned}
$$

we also know that invasion criteria is a quadraticpolynomial of δ, which by mean value theorem means that there is only one zero for each value of p in interval $\delta \in (0,1)$. Figures 3.5 and 3.6 show how zero line moves in parameter space when ratio V/C changes.

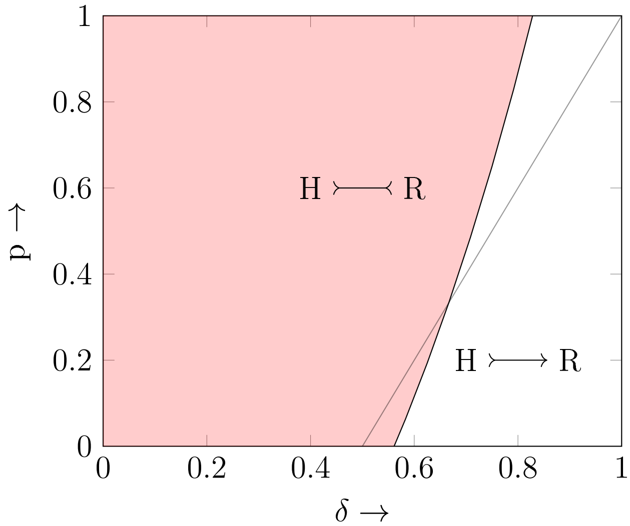

Figure 3.5: Hawk and Retaliator invasion when $\frac{V}{C} < 1$

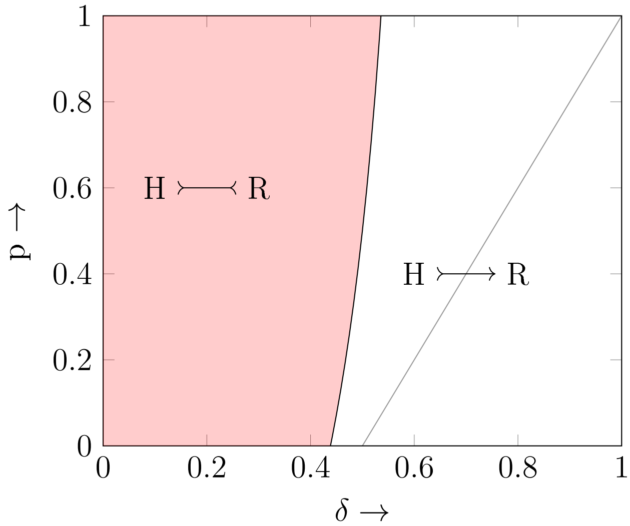

Figure 3.6: Hawk and Retaliator invasion when $\frac{V}{C} > 1$

3.7 Bully and Retaliator invasibility

Now assume Retaliator as resident strategy and Bully as the invading strategy. Now $S_r(0) \approx 1$ and invading $S_b(0)$ is small. Invasion criteria is

$$\begin{aligned}
E_b - E_r = V + \delta\frac{V-C}{2} + \frac{V}{2}\frac{\delta^3}{1-\delta} - \frac{1+p\;r(b,r)}{1+p\;r(r,r)}\frac{V}{2}\frac{1}{1-\delta} &> 0 \\
V + \delta(-V-C) + \delta^2(-V+C) + \delta^3 V &> 0 \\
(\delta-1)(\delta^2 + \delta\frac{C}{V} - 1) &> 0 \\
\delta^2 + \delta\frac{C}{V} - 1 &< 0,
\end{aligned}$$

which, by mean value theorem, has a zero for $\delta \in (0,1)$.

Now assume Bully as resident strategy and Retaliator as the invading strategy. Now $S_b(0) \approx 1$ and invading $S_r(0)$ is small. Invasion criteria is

$$\begin{aligned}
E_r - E_b = \delta\frac{V-C}{2} + \delta^2\frac{V}{2} + \frac{V}{2}\frac{\delta^3}{1-\delta} - \frac{1+p\;r(b,r)}{1+p\;r(b,b)}\frac{V-C}{2} + \frac{V}{2}\frac{\delta}{1-\delta} &> 0 \\
(1-\delta)^2(V(1+\delta) - C) &> 0 \\
V + \delta V - C &> 0 \\
\delta &> \frac{C}{V} - 1.
\end{aligned}$$

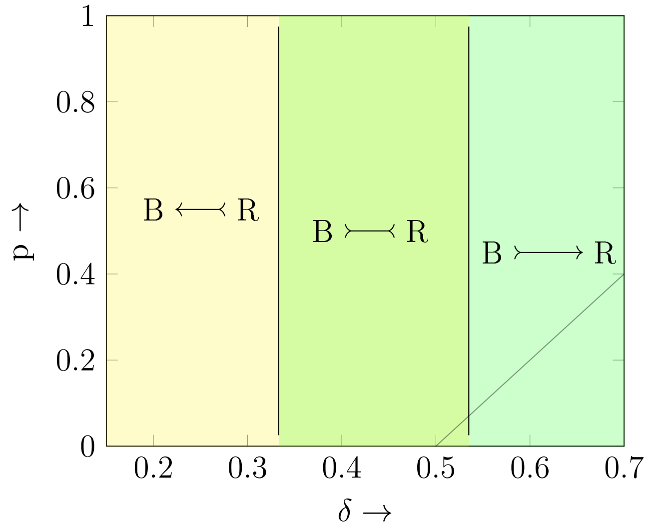

Figure 3.7: Pairwise invasions for Bully and Retaliator strategies when $V < C < 1.5V$

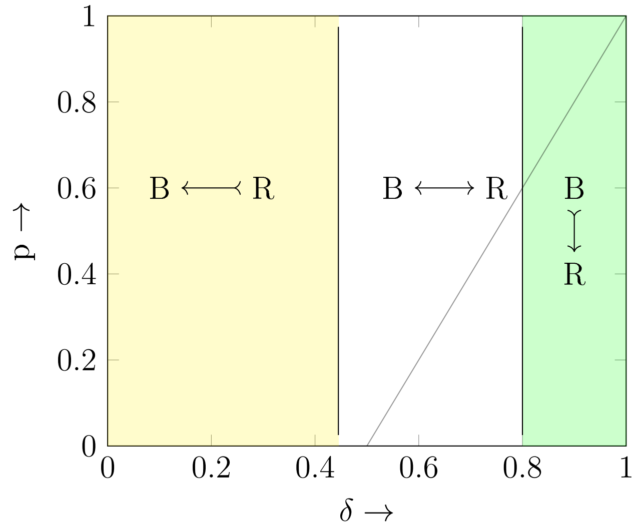

Figure 3.8: Pairwise invasions for Bully and Retaliator strategies when $1.5V < C < 2V$

3.8 Pairwise invasibility graphs for pooled games

None of the invasion criteria in previous section depend on difference $V - C$ so we can without loss of generality assume that $V = 1$ and vary value of C. I plot parameter space p, δ for $C = \{0.5, 1.4, 1.47, 1.51, 1.75, 1.83, 3\}$ to show all possible configurations for the P.I.G.s. If an area in the plot is unmarked, it is included in previous plots. Pairwise Invasibility Graphs have ESS strategies marked in Blue, in case of Retaliator and Dove mix of those strategies is ESS. If only one of the Dove and Retaliator is marked with blue, it means that strategy is ESS, but non-rare influx of the other strategy can cause the population to become invadable by Hawk or Bully. See chapter 3 for structure of PIGs.

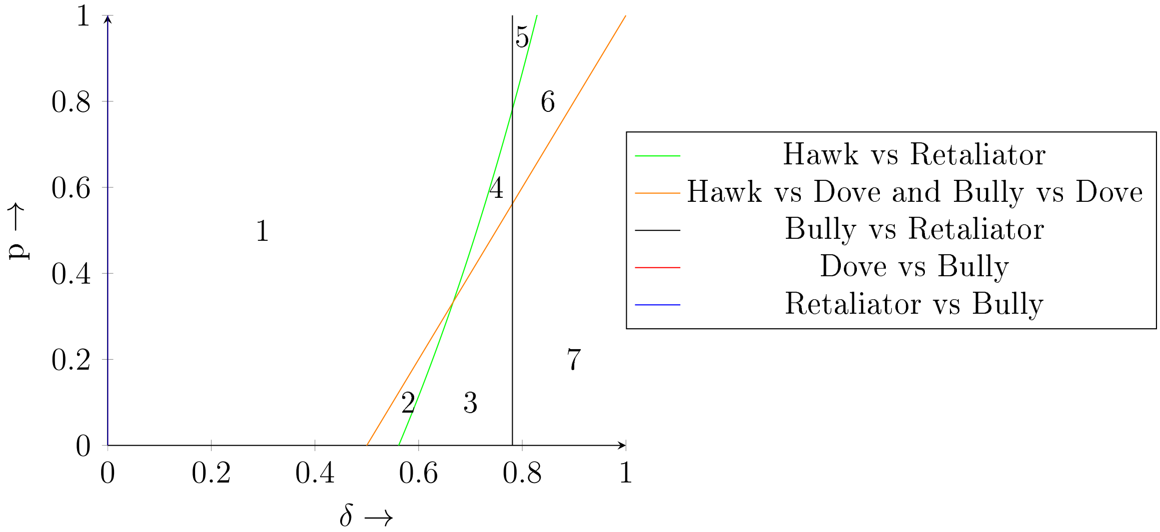

Figure 3.9: C = 0.5. Each curve represents bifurcation parameter values where invasibility changes. Invader is written on the left and resident on the right. See chapter 3 for more detail on each line.

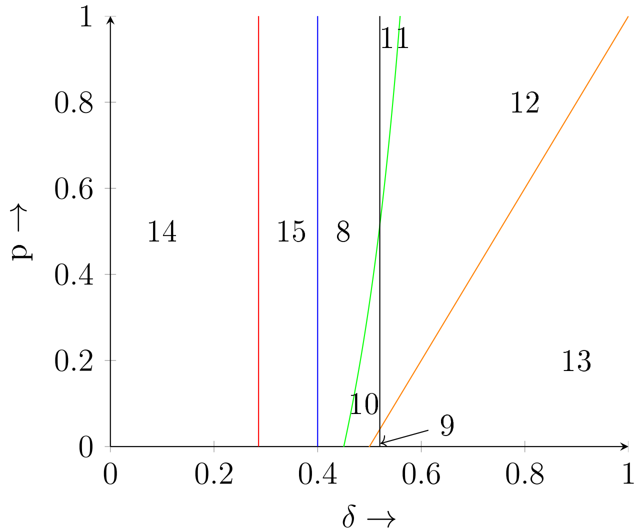

Figure 3.10: C = 1.4, see figure 3.9

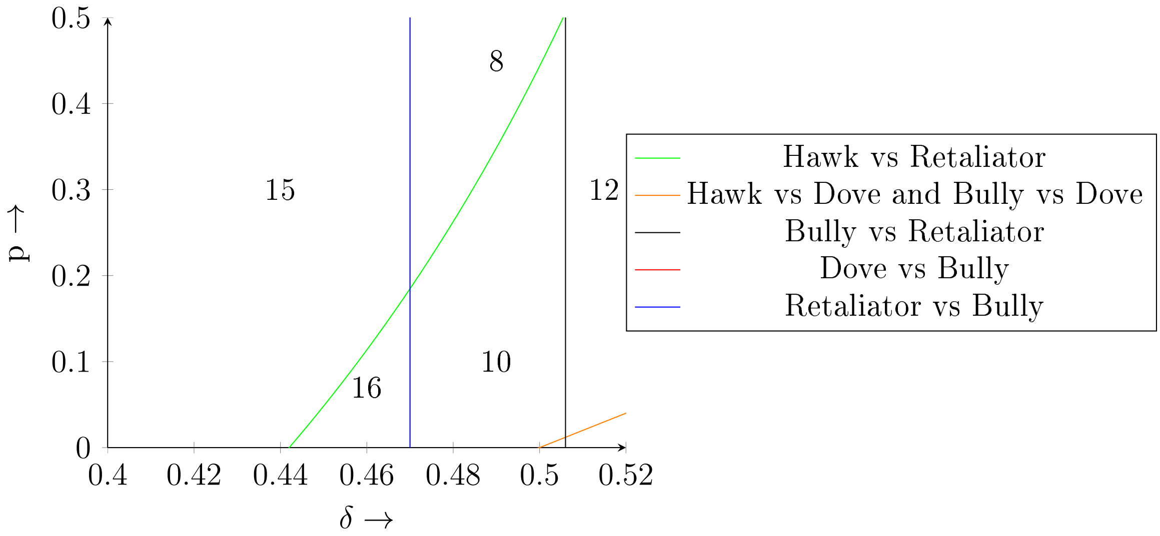

Figure 3.11: C = 1.47, see figure 3.9

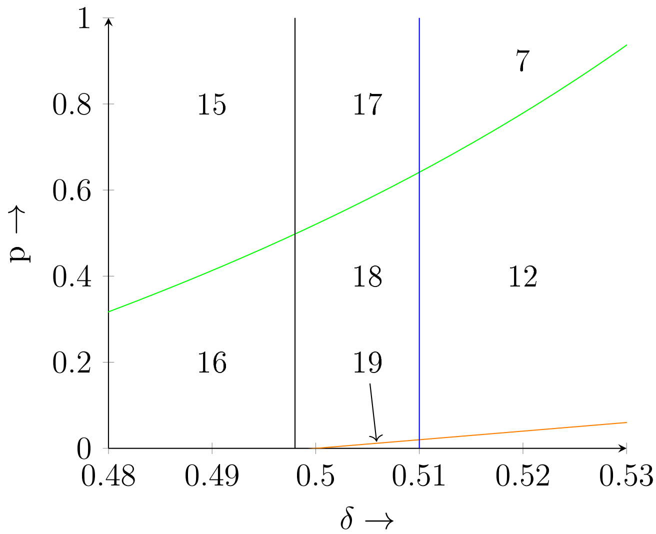

Figure 3.12: C = 1.51, see figure 3.9

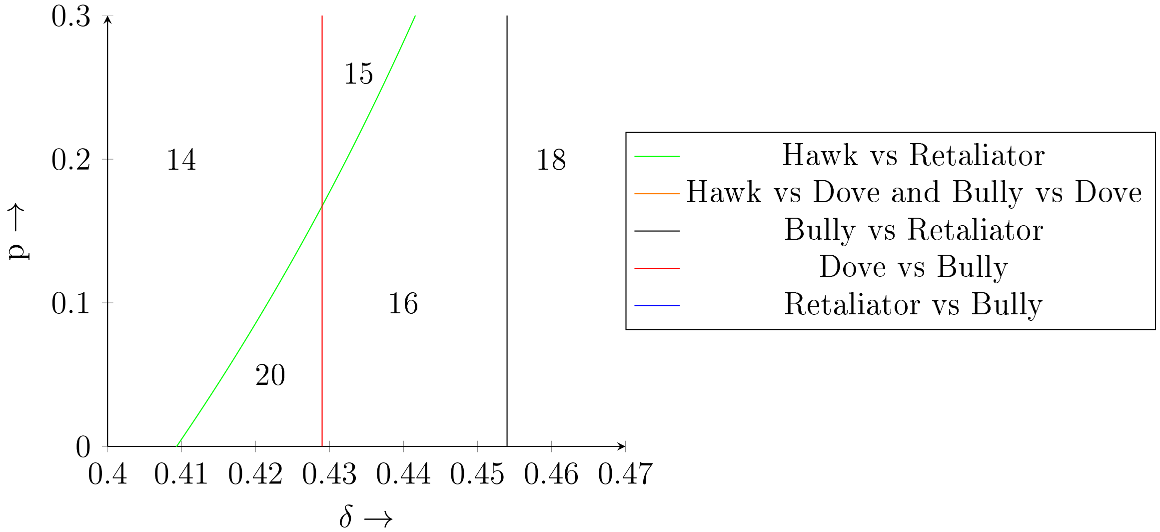

Figure 3.13: C = 1.75, see figure 3.9

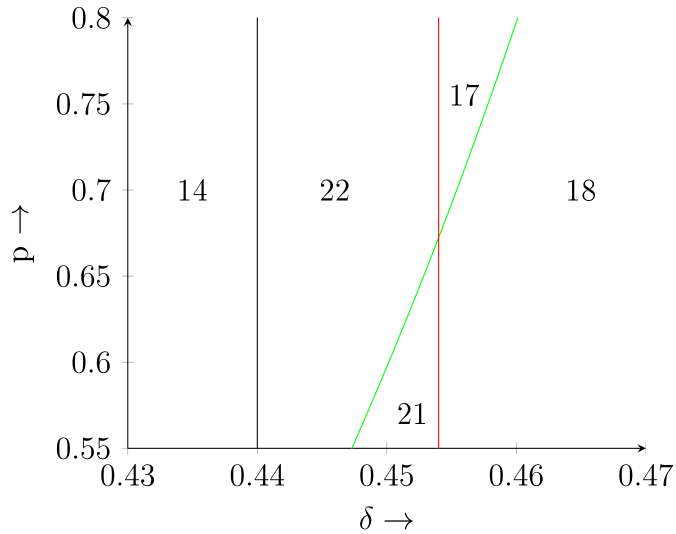

Figure 3.14: C = 1.86, see figure 3.9

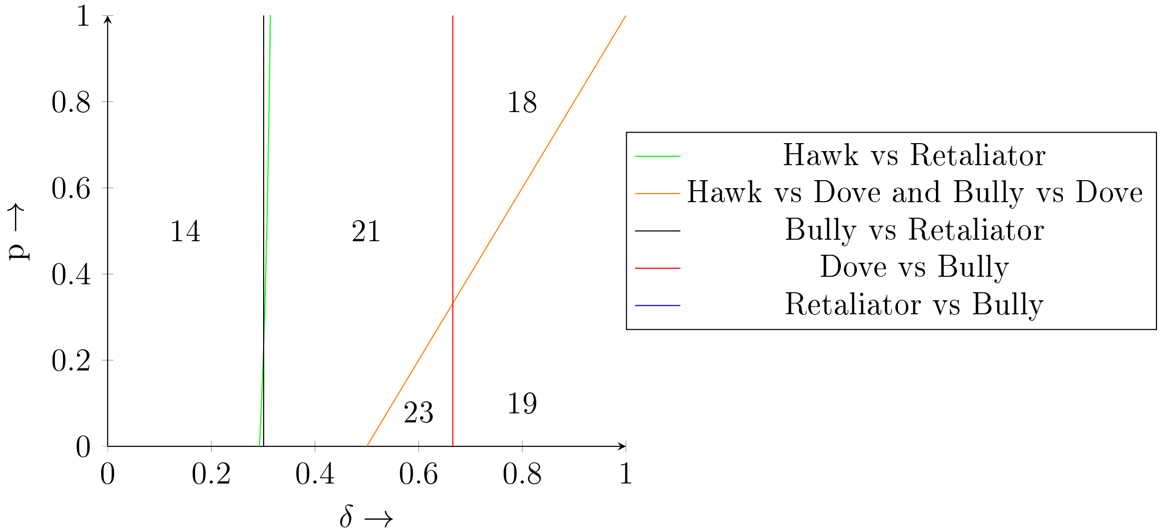

Figure 3.15: C = 3, see figure 3.9

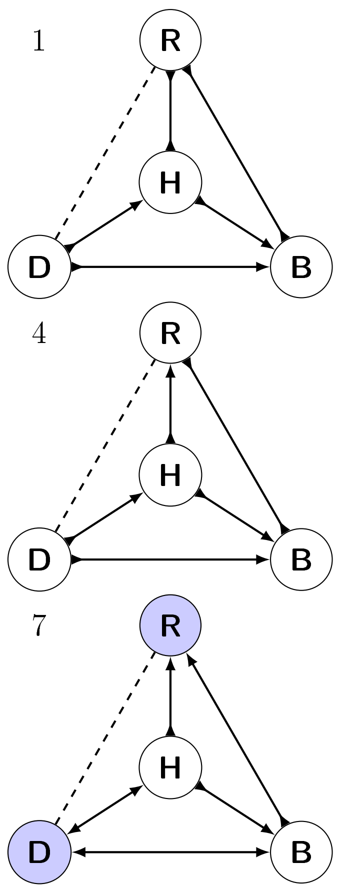

Figure 3.16: First seven possible Pairwise invasibility graphs, numbering corresponding to figure 3.9.

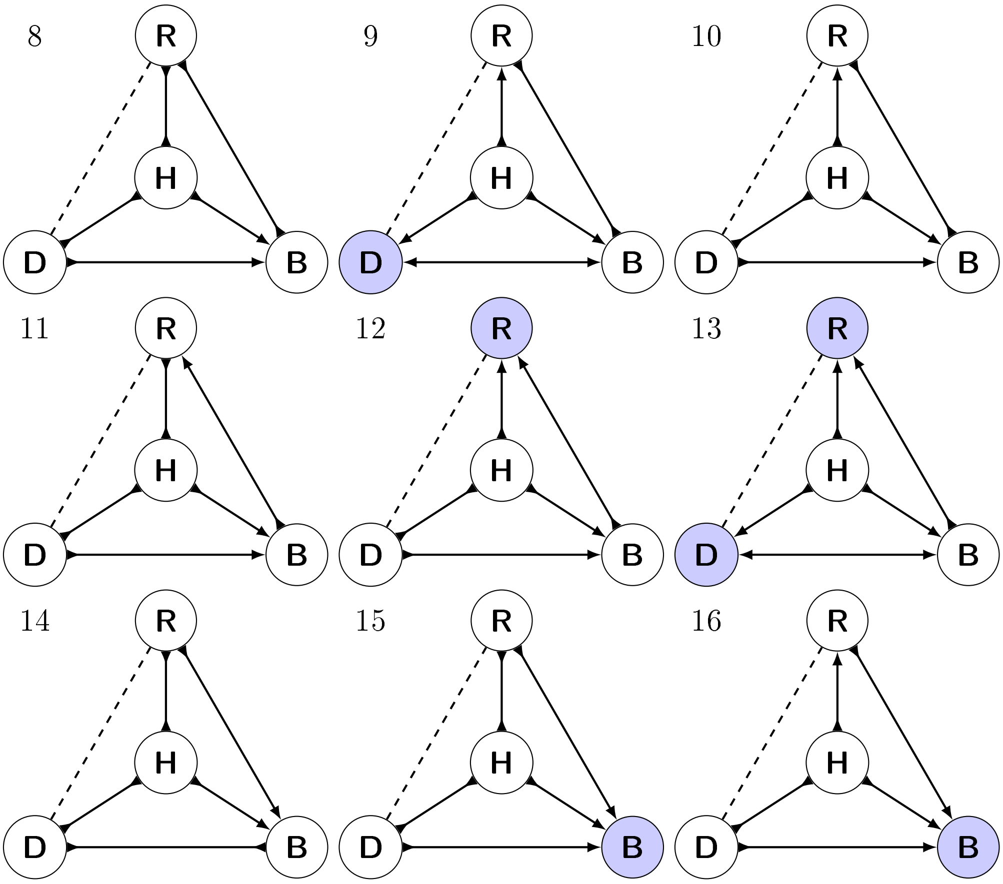

Figure 3.17: Nine possible Pairwise invasibility graphs, numbering corresponding to figures 3.12, 3.13, 3.14 and 3.15

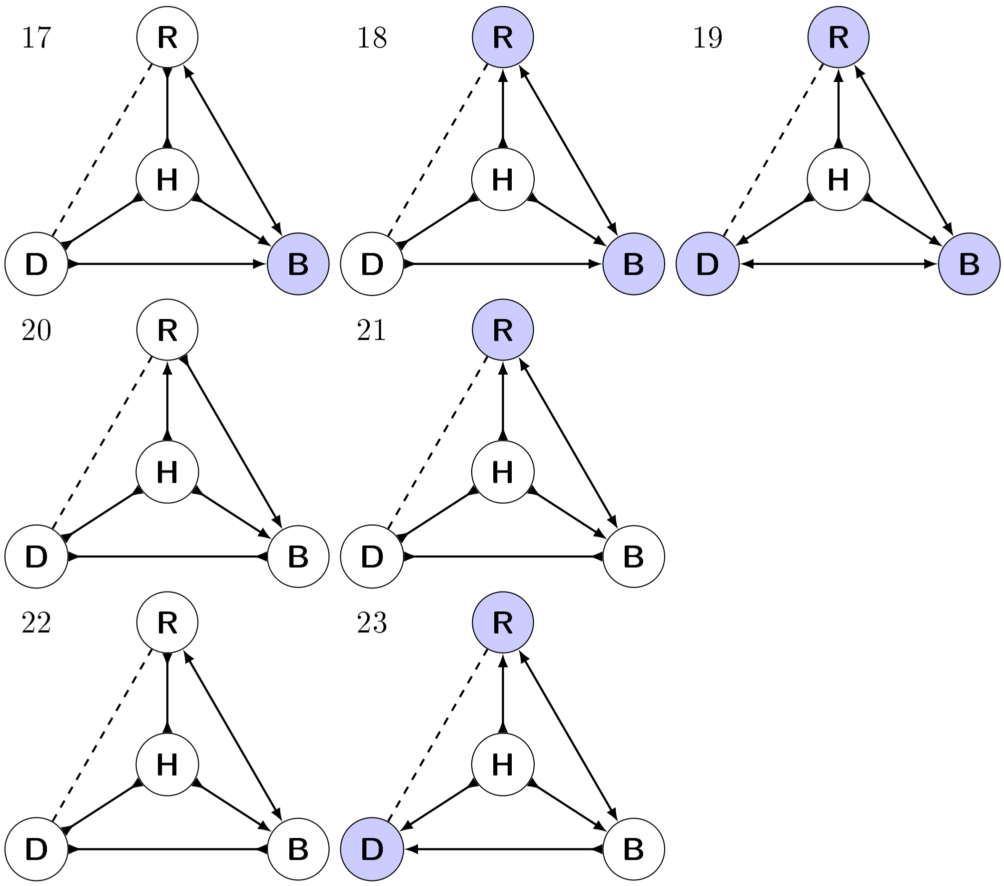

Figure 3.18: Last seven possible (out of 23 total) Pairwise invasibility graphs, numbering corresponding to figures 3.10, 3.11, 3.12, 3.13, 3.14 and 3.15

Chapter 4

Discussions

4.1 Results

Pooled games add pooling and quitting rule to iterated games. Some pairwise invasibility pairs are possible without pooled games, but many PIGs generated in this thesis are not, most notably those PIGs that require quitting for viability. In iterated games without quitting, Hawk can always invade Dove. In contrast pooled games show that Dove population can prevent Hawk invasion by using quitting rule. Quitting early against Hawk, Doves make few invading Hawks pariahs, that play too rarely to gain payoffs for invasion. Comparing single round PIGs 2.7 and 2.8 to pooled games PIGs the difference in possible configurations becomes even more clear.

If each strategy has same quitting round, pooled game invasion criteria between those strategies are equivalent to single iterated game. This can be seen from equation 1.24. If f_{ij} is constant for all i and j, the ratio $x/x_0 = 1$. This means that pooled games require quitting as part of strategies to be different from regular single iterated games.

In iterated games with quitting rule but without pooling, many pairwise invasions of pooled games are possible. However some PIGs are not possible, for example figure 3.17 has PIG number 11. This Pairwise Invasibility Graph is not possible without pooling, more precisely Hawk and Retaliator coexist with Retaliator being able to invade Bully and Bully not being able to invade Retaliator. Individually each pair is possible without pooled games, but to have each in same PIG requires to have both pooling and quitting at the same time. This is because without pooling parameter space is $\{\delta\}$ and with pooled games parameter space is $\{\delta, \text{p}\}$.

Quitting strategies in this form assume that playing first round of the game is either unavoidable or trying to play is desirable in some way. It is easy to see that if a player keeps playing game where it loses resources each time, it would be better for it to quit instead. Assuming resources that are not accessible by individuals, but are by pair of individuals, Hawk-Dove version of this game might be about what happens to the resource after it has been gathered. This sort of competition might be basis for quitting in middle of the game, despite there still being resource to be gathered.

4.2 Extensions

4.2.1 Meta-strategies

We can allow strategies to have meta-strategies, e.g. each meta-strategy would choose (randomly or otherwise) their strategy at the start of each game. Players would not change their strategies in between rounds, only between games. This would break each game into multiple parts and adds some calculations, but as I defined the pooled game, it allows for such generalization. These meta-strategies are no explored further in this thesis.

4.2.2 Non-playing income

One thing I did not take into account is the effect of background income or non-playing strategys income to the equilibrias and invasions. If we allow each strategy to have payoff α for each round they are not playing, sign equivalent invasion fitness then becomes

(4.1)
$$E_2 - E_1 \overset{sign}{=} p(x(\sum_{h}^{f_{21}} a_{21}^h - \frac{x}{x_0}\sum_{h}^{f_{11}} a_{11}^h) + (\sum_{h}^{f_{22}} a_{22}^h - \frac{x}{x_0}\sum_{h}^{f_{12}} a_{12}^h)) + \alpha(1+x)(1-\frac{x}{x_0}),$$

which for $\alpha = 0$ gives pooled games as studied in this thesis.

4.2.3 Evolution of quitting rules

I defined HDRB-game to have fixed quitting rules. In 3 hawk quits immediately after facing another hawk, but if instead it quit with probability τ, which way would the parameter evolve and in which populations? Lets define hawk versus hawk contest as one getting hurt and the other gaining whole the reward. If in pooled game Hawk instead of quitting anytime it faced a hawk, it instead quit only if it did not get the reward. It would be interesting to see which strategy fares better and in which conditions.

Pablov is a strategy that switches what it plays each time it gets too low payoff for the round. A quitting rule for this kind of strategy would probably be if its total payoff is too low it quits. How high or low should each limit be?

In this thesis Bully learns its lesson after first hawk is played against it. Sneaky bully strategy is where after n rounds Sneaky bully plays hawk to test its opponent. Evolving quitting strategy on it could be about how many times it plays hawk.

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